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Viscous incompressible flow between concentric rotating spheres. Part 2. Hydrodynamic stability

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The energy theory of hydrodynamic stability is applied to the viscous incompressible flow of a fluid contained between two concentric spheres which rotate about a common axis with prescribed angular velocities. The critical Reynolds number is calculated for various radius and angular velocity ratios such that it is certain the basic laminar motion is stable to any disturbances. The stability problem is solved by means of a toroidal–poloidal representation of the disturbance flow and numerical integration of the resulting eigenvalue problem.

1. Introduction

We consider here the stability of the steady laminar motion of a viscous incompressible fluid contained between concentric rotating spheres that rotate about a common axis. This basic laminar motion is discussed in part 1 (Munson & Joseph 1971) and the geometry of the spherical annulus is shown in figure 1. Aside from the desire to know the critical Reynolds number, the stability consideration of this basic flow is of interest, on its own, for the following reasons: (*a*) the flow field is completely bounded, unlike, for example, Poiseuille or Couette flow, and (*b*) the character of the basic flow is dependent on the Reynolds number, again unlike Poiseuille or Couette flow.

The energy (stability) problem is considered for the various basic flow cases considered in part 1 and the critical Reynolds number, ρ_E, is determined according to the energy theory, so that with $Re < \rho_E$ it is certain that the basic flow is stable to any disturbances (large or small). The Reynolds number, Re, is defined as $Re = \Omega_0 R_2^2/\nu$, where R_2, the radius of the outer sphere, is taken as the characteristic length and Ω_0 the characteristic angular velocity. In general Ω_0 will be either Ω_1 or Ω_2 (the constant angular velocity of the inner or outer sphere, respectively) depending on the relative influence of either sphere for the particular situation being considered.

The energy theory, which has its beginnings with Reynolds (1895) and Orr (1907) around the turn of the century, has in recent times been greatly expanded by Serrin (1959) and Joseph (1966). The method consists of considering the time rate of change of the kinetic energy of an arbitrary disturbance in the flow field. If the kinetic energy decreases as a function of time, we say the flow is stable.

Bratukhin (1961) obtained an approximate linear stability result for flow between spheres with the outer sphere stationary and the radius ratio $\eta \equiv R_1/R_2 = 0{\cdot}5$. Although he used the Stokes flow solution (lowest-order perturbation solution) as the basic flow and obtained only an approximate solution to the linear problem, the results tend to support the idea that the linear and energy results for this sphere problem may be quite near one another. Hence the region of possible sublinear instability may be small.

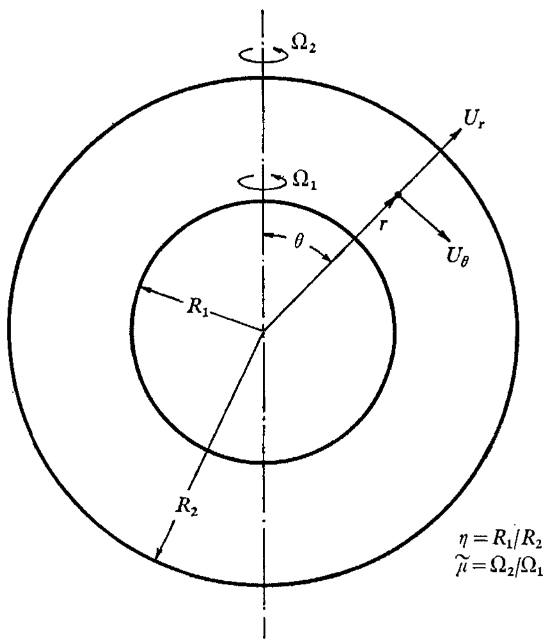

FIGURE 1. Spherical annulus.

Because the basic flow between spheres is a function of two spatial variables, r, θ, the eigenvalue problem associated with the energy theory is a partial differential equation which cannot be reduced to an ordinary differential equation via the customary use of normal modes. This fact combined with the condition that the spherical annulus is a completely bounded region dictates a somewhat different approach to the solution of the stability eigenvalue problem. Various convection instability problems (energy or linear theory) have been considered for bounded domains such as spherical shells (Chandrasekhar 1961, Joseph & Carmi 1966), a rectangular box (Davis 1967), and other geometries in the papers of Zierep (1963) and Ostrach & Pnueli (1963). A major difference between these convection considerations and the present shear flow between spheres is that, unlike the quiescent fluid for the 'basic flow' of the convection problems, the basic flow between the spheres is a function of two spatial variables, r, θ, and is strongly dependent on the Reynolds number, Re.

2. Energy theory for stability of a viscous flow

We consider the energy stability analysis of the basic laminar flow between concentric rotating spheres which was obtained in part 1. This general stability theory has its beginnings with Reynolds (1895) and Orr (1907) and essentially considers the time rate of change of the kinetic energy of an arbitrary disturbance in the flow field. If the kinetic energy always decreases as a function of time, we say that the flow is stable. In recent times Serrin (1959) and Joseph (1966) have successfully extended the method and applied it to various problems. Included are a universal stability criterion for any bounded flow region and the incorporation of temperature-driven buoyancy forces, in terms of the Boussinesq equations. Other papers in which the energy method has been used for specific basic flows include those by Joseph & Carmi (1969), Shir & Joseph (1968) and Joseph & Munson (1970).

Whereas the linear theory of hydrodynamic stability considers only infinitesimal disturbances and provides a critical Reynolds number, ρ_L, such that the basic flow is definitely unstable for $Re > \rho_L$, it can say nothing concerning the possibility of instabilities caused by disturbances of finite size. On the other hand, the energy theory considers any size disturbance (large or small) and provides a critical Reynolds number, ρ_E, such that the flow is definitely stable to all disturbances if $Re < \rho_E$. The energy theory, however, cannot predict instability – only stability.

In order to obtain the energy (stability) equations, we consider the following. Let $\mathbf{U}(\mathbf{r})$ and $\mathbf{u}(\mathbf{r})$ denote the basic laminar flow and the arbitrary disturbance, respectively. The non-linear Navier–Stokes equations then become

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\,.\nabla)\,\mathbf{u}+(\mathbf{u}\,.\nabla)\,\mathbf{U}+(\mathbf{U}\,.\nabla)\,\mathbf{u} = -\nabla p+\frac{1}{Re}\nabla^2\mathbf{u}, \tag{1}$$

$$\nabla\,.\,\mathbf{u} = 0, \quad \mathbf{r}\in\mathscr{V}, \tag{2}$$

with boundary conditions

$$\mathbf{u} = 0, \quad \mathbf{r}\in\partial\mathscr{V}.$$

Let $\langle\,\rangle$ designate integration over the volume between the two spheres. Multiplication of (1) by $\mathbf{u}$. followed by $\langle\,\rangle$ gives the following (see Serrin 1959),

$$\frac{1}{2}\frac{d}{dt}\langle|\mathbf{u}|^2\rangle+\langle\mathbf{u}\,.\nabla\mathbf{U}\,.\mathbf{u}\rangle = -\frac{1}{Re}\langle\nabla\mathbf{u}\colon\nabla\mathbf{u}\rangle. \tag{3}$$

Thus $$\frac{1}{2}\frac{d}{dt}\langle|\mathbf{u}|^2\rangle = \langle|\nabla\mathbf{u}|^2\rangle\left\{-\frac{\langle\mathbf{u}\,.\nabla\mathbf{U}\,.\mathbf{u}\rangle}{\langle|\nabla\mathbf{u}|^2\rangle}-\frac{1}{Re}\right\} \leqslant a^2\left(\frac{1}{\rho_E}-\frac{1}{Re}\right)\langle|\mathbf{u}|^2\rangle,$$

as long as $Re < \rho_E$. Here $$\langle|\nabla\mathbf{u}|^2\rangle \geqslant a^2\langle|\mathbf{u}|^2\rangle,$$

$$\frac{1}{\rho_E} = \max_{h}\{-\langle\mathbf{u}\,.\nabla\mathbf{U}\,.\mathbf{u}\rangle/\langle|\nabla\mathbf{u}|^2\rangle\}, \tag{4}$$

where h is the collection of smooth functions satisfying $\nabla\,.\,\mathbf{u} = 0$ and (2). Then

$$\tfrac{1}{2}\langle|\mathbf{u}|^2\rangle \leqslant \tfrac{1}{2}\langle|\mathbf{u}|^2\rangle|_{t=0}\exp\left\{2a^2\left(\frac{1}{\rho_E}-\frac{1}{Re}\right)t\right\}. \tag{5}$$

It follows that if $Re < \rho_E$ the motion $\mathbf{U}$ is stable and any disturbance of this motion decays very fast. We obtain the energy limit, ρ_E, as the smallest of the eigenvalues of the Euler equations of the functional given in (4). These are,

$$\mathscr{D}.\mathbf{u} = (1/\rho)\nabla^2\mathbf{u} - \nabla p, \qquad (6)$$

$$\nabla.\mathbf{u} = 0$$

and $\mathbf{u} = 0$ on the boundaries. Here $\mathscr{D}$ is the rate of strain matrix – the symmetric part of $\nabla\mathbf{U}$.

The task is to solve the eigenvalue problem given in (6) for the least eigenvalue, $\rho = \rho_E$, where $\mathscr{D}$ is given by the basic flow as determined in part 1. The present problem, flow between rotating spheres, is different from the previously studied problems in regard to the following. Usually the basic flow considered is independent of the Reynolds number. For example, plane Couette flow and Poiseuille flow consist of a linear shear or parabolic profile, each of which (in non-dimensional form) is independent of Reynolds number. Hence, the rate of strain matrix, $\mathscr{D}$, is independent of Re so that the Reynolds number appears (both explicitly and implicitly) only as the coefficient of the $\nabla^2\mathbf{u}$ term in (6). On the other hand, as

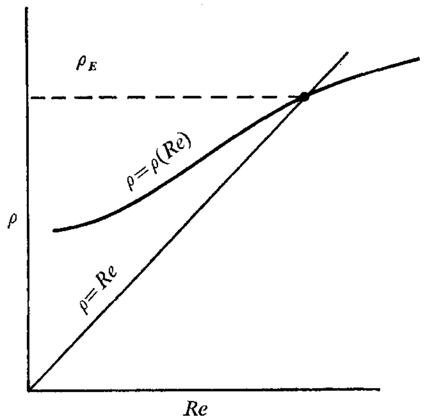

FIGURE 2

seen in part 1 the non-dimensional basic flow between rotating spheres is strongly dependent on the Reynolds number. Thus, it is not possible to consider 'a specific flow' for all Reynolds numbers and determine the critical Reynolds number directly from it. Rather we must consider the critical Reynolds number, ρ_E, as a function of the basic flow Reynolds number, Re, as follows. We must determine ρ_E so that it corresponds precisely with the Reynolds number of the basic flow being considered. That is, we want the intersection of the $\rho = \rho(Re)$ curve with $\rho = Re$ as indicated in figure 2. For $Re < \rho_E$ the fact that $\rho > Re$ shows that this basic flow is stable, while for $Re > \rho_E$ the fact that $\rho < Re$ shows that the critical value is lower than the value of Re being considered. The intersection of these two curves gives the critical Reynolds number, ρ_E, for the particular geometry being considered. Hence for a given radius ratio, η, and a given ratio of angular velocities,

$\tilde{\mu} = \Omega_2/\Omega_1$, we must determine the curve $\rho = \rho(Re)$ of the lowest eigenvalue of (6). The critical value where $\rho(Re) = Re$ (denoted ρ_E) then indicates the value of Re below which it is certain that the motion is stable to any disturbances.

3. Governing equations for energy theory in toroidal–poloidal representation

We must solve the eigenvalue problem given by (6) where the rate of strain matrix, $\mathscr{D}$, has components

$$\left.\begin{aligned} d_{rr} &= \frac{\partial U}{\partial r}, \quad d_{\theta\theta} = \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{U}{r}, \quad d_{\phi\phi} = \frac{U}{r} + \coth\theta\frac{V}{r}, \\ d_{r\theta} &= d_{\theta r} \qquad = \frac{1}{2r}\frac{\partial U}{\partial \theta} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{V}{r}\right), \\ d_{\theta\phi} &= d_{\phi\theta} = \frac{\sin\theta}{2r}\frac{\partial}{\partial\theta}\left(\frac{W}{\sin\theta}\right), \quad d_{r\phi} = d_{\phi r} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{W}{r}\right). \end{aligned}\right\} \tag{7}$$

In the above the basic laminar flow is written as $\mathbf{U} = (U_r, U_\theta, U_\phi) = (U, V, W)$, where, in terms of the Legendre polynomial series representation given in part 1, we find that

$$\left.\begin{aligned} U &= \sum_s \left(\frac{g_s}{r^2}\right)[2\cos\theta\, P_s + \sin\theta\, \dot{P}_s], \\ V &= \sum_s \left(-\frac{g_s'}{r}\right)\sin\theta\, P_s, \\ W &= \sum_s \left(\frac{f_s}{r}\right)\sin\theta\, P_s, \end{aligned}\right\} \tag{8}$$

where $(\;)' \equiv d/dr$ and $(\dot{\;}) \equiv d/d\theta$. Recall that the basic flow is independent of the longitude, ϕ. The basic flow component functions $f_s(r)$, $g_s(r)$, are obtained in part 1 by a high-order perturbation result or by a numerical technique.

One difficulty in solving this problem is that because the basic flow is a function of two spatial co-ordinates the governing equations are partial differential equations. The more common situation is that the basic flow is a function of one spatial co-ordinate (as in Couette or Poiseuille flow or the similarity variable of boundary layers or Jeffery–Hamel flow) which allows the introduction of two wave-numbers and the reduction of the partial differential equations to ordinary differential equations. In the present case because the basic flow is independent of ϕ, a wave-number in the ϕ direction, m, can be used, but the two independent variables r, θ still remain. The second difficulty in solving (6) stems from the fact that, because of its complexity, the basic flow is known only numerically as discussed in part 1. Thus, the eigenvalue problem is solved by appropriate series representation and numerical integration.

Whereas a stream function and angular velocity function are used in the determination of the axially symmetric basic flow, similar functions cannot be used for the disturbance flow calculations because the possibility of non-axisymmetric disturbances must be considered. In fact the critical eigenvalue is given by a non-axisymmetric disturbance motion with $m = 1$.

However, the partial differential equations can be reduced to an equivalent infinite set of coupled linear equations by taking advantage of the incompressibility condition, $\nabla . \mathbf{u} = 0$, and making an appropriate series representation. As discussed in Chandrasekhar (1961) any solenoidal vector, $\mathbf{u}$, can be written in terms of two vector components, a toroidal component, $\mathbf{T}$, and a poloidal component, $\mathbf{S}$, as

$$\mathbf{u} = \mathbf{T} + \mathbf{S}, \tag{9}$$

where $\mathbf{T}$ and $\mathbf{S}$ are defined by their generating scalars Ψ, Φ as follows:

$$\mathbf{T} = \text{curl}\left(\frac{\Psi}{r}\mathbf{r}\right), \quad \mathbf{S} = \text{curl}\left[\text{curl}\left(\frac{\Phi}{r}\mathbf{r}\right)\right]. \tag{10}$$

It is then possible to expand the generating scalars in terms of spherical harmonics as

$$\left.\begin{aligned} \Psi &= \sum_{l,m} T_l(r)\, Y_l^m(\theta,\phi), \\ \Phi &= \sum_{l,m} S_l(r)\, Y_l^m(\theta,\phi), \end{aligned}\right\} \tag{11}$$

where $Y_l^m(\theta,\phi)$ is the spherical harmonic defined by

$$Y_l^m(\theta,\phi) = e^{im\phi} P_l^{|m|}(\cos\theta) \tag{12}$$

and $P_l^{|m|}(\cos\theta)$ are the associated Legendre polynomials defined as

$$P_l^m(x) = (1-x^2)^{\frac{1}{2}m}\,(d/dx)^{l+m}\frac{(x^2-1)^l}{2^l l!}, \quad x = \cos\theta. \tag{13}$$

Since the basic flow is independent of ϕ we can consider a specific value of m, the axial wave-number, (rather than include a double sum over l and m) and determine the minimum eigenvalue over integer values of m as usual. With this treatment we have the disturbance velocity represented as

$$\mathbf{u} = \sum_l \left\{\frac{l(l+1)}{r^2} S_l Y_l^m, \quad \frac{1}{r} S_r' \dot{Y}_l^m + \frac{im}{r\sin\theta} T_l Y_l^m, \quad \frac{im}{r\sin\theta} S_l' Y_l^m - \frac{1}{r} T_l \dot{Y}_l^m\right\}. \tag{14}$$

The task, then, is to write the governing equations (6) in terms of the disturbance flow component functions, $T_l(r), S_l(r)$, truncate the system at some appropriate value, $l = L_t$, and solve this system of equations. Sherman (1968) has used the toroidal–poloidal representation with a Galerkin approximation in dealing with convective flow within a sphere.

The system of equations governing the disturbance flow component functions, T_l, S_l, is obtained as indicated below. More detail of their derivation can be obtained from Munson (1970). To eliminate the 'pressure', p, we take the curl of (6) and obtain

$$\text{curl}^3\,\mathbf{u} + \rho\,\text{curl}\,(\mathscr{D}.\mathbf{u}) = 0. \tag{15}$$

We substitute the toroidal–poloidal representation given by (14), multiply by appropriate functions, $\tilde{S}_{l'}^{m'}$, $\tilde{T}_{l'}^{m'}$, where the defining scalar for these vectors is unity, and integrate over the unit sphere. We use various orthogonal properties of toroidal–poloidal vectors discussed in Chandrasekhar (1961) to obtain the following set of coupled linear ordinary differential equations:

$$\hat{L}_l^2 T_l - \rho M_l^m \sum_n \int_0^\pi Y_l^{-m} \mathscr{F}_{r_n}^m \sin\theta \, d\theta = 0, \tag{16}$$

$$\hat{L}_l^4 S_l - \rho M_l^m \sum_n \left[\frac{im}{r}\int_0^\pi Y_l^{-m}\mathscr{F}_{\theta_n}^m \, d\theta + \frac{1}{r}\int_0^\pi \dot{Y}_l^{-m}\mathscr{F}_{\phi_n}^m \sin\theta\, d\theta\right] = 0, \tag{17}$$

where
$$\hat{L}_l^2 = d^2/dr^2 - l(l+1)/r^2, \tag{18}$$

$$M_l^m = \frac{2l+1}{2l(l+1)}\frac{(l-|m|)!}{(l+|m|)!}.$$

The corresponding homogeneous boundary conditions are

$$S_l = S_l' = T_l = 0 \quad \text{for} \quad r = \eta, \quad r = 1. \tag{19}$$

In the above
$$\mathscr{F}_n^m \equiv \{\mathscr{F}_{r_n}^m, \mathscr{F}_{\theta_n}^m, \mathscr{F}_{\phi_n}^m\} = r^2 \,\mathrm{curl}\,[\mathscr{D}\,.\,(\mathbf{T}_n + \mathbf{S}_n)]. \tag{20}$$

After considerable algebra the integrals in (16), (17) can be evaluated and the equations written in the condensed form

$$\hat{L}_{l-1}^2 T_{l-1} - \rho A_l^m \sum_{n=m+1}^{\infty} [G_{1_{nl}} S_{n-1}' + G_{2_{nl}} S_{n-1} + G_{3_{nl}} T_{n-1}] = 0, \tag{21}$$

$$\hat{L}_{l-1}^4 S_{l-1} - \rho A_l^m \sum_{n=m+1}^{\infty} [H_{1_{nl}} S_{n-1}'' + H_{2_{nl}} S_{n-1}' + H_{3_{nl}} S_{n-1} + H_{4_{nl}} T_{n-1}' + H_{5_{nl}} T_{n-1}] = 0$$

$$\text{for} \quad l = m+1, m+2, \ldots, \tag{22}$$

where
$$A_l^m = \frac{(2l-1)}{2l(l-1)}\frac{(l-m-1)!}{(l+m-1)!}. \tag{23}$$

We have used the fact that $T_l(r) = S_l(r) \equiv 0$ for $l < m$ and $T_0(r) = S_0(r) \equiv 0$ for $m = 0$ which follows from the condition that $P_l^m = 0$ for $l \leqslant m$. The variable coefficients $G_{1_{nl}}(r)$, $G_{2_{nl}}(r), \ldots, H_{5_{nl}}(r)$ are functions of the basic flow through the component functions $f_s(r)$, $g_s(r)$ as follows:

$$\left.\begin{aligned}
G_{1_{nl}} &= \frac{1}{r^2}\sum_{s=1}^{N_t+1}(\alpha_1 f_{s-1} + i\alpha_2 g_{s-1}' + i\alpha_3 g_{s-1}/r),\\
G_{2_{nl}} &= \frac{1}{r^2}\sum_{s=1}^{N_t+1}(\alpha_4 f_{s-1}' + \alpha_5 f_{s-1}/r + i\alpha_6 g_{s-1}'' + i\alpha_7 g_{s-1}'/r + i\alpha_8 g_{s-1}/r^2),\\
G_{3_{nl}} &= \frac{1}{r^2}\sum_{s=1}^{N_t+1}(i\alpha_9 f_{s-1} + \alpha_{10} g_{s-1}' + \alpha_{11} g_{s-1}/r),
\end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned}
H_{1_{nl}} &= \frac{1}{r^2}\sum_{s=1}^{N_t+1}(i\beta_1 f_{s-1} + \beta_2 g_{s-1}' + \beta_3 g_{s-1}/r),\\
H_{2_{nl}} &= \frac{1}{r^2}\sum_{s=1}^{N_t+1}(i\beta_4 f_{s-1}' + i\beta_5 f_{s-1}/r + \beta_6 g_{s-1}'' + \beta_7 g_{s-1}'/r + \beta_8 g_{s-1}/r^2),\\
H_{3_{nl}} &= \frac{1}{r^2}\sum_{s=1}^{N_t+1}(i\beta_9 f_{s-1}'' + i\beta_{10} f_{s-1}'/r + i\beta_{11} f_{s-1}/r^2 + \beta_{12} g_{s-1}''' + \beta_{13} g_{s-1}''/r\\
&\qquad + \beta_{14} g_{s-1}'/r^2 + \beta_{15} g_{s-1}/r^3),\\
H_{4_{nl}} &= \frac{1}{r^2}\sum_{s=1}^{N_t+1}(\beta_{16} f_{s-1} + i\beta_{17} g_{s-1}' + i\beta_{18} g_{s-1}/r),\\
H_{5_{nl}} &= \frac{1}{r^2}\sum_{s=1}^{N_t+1}(\beta_{19} f_{\ -1} + \beta_{20} f_{s-1}/r + i\beta_{21} g_{s-1}'' + i\beta_{22} g_{s-1}'/r + i\beta_{23} g_{s-1}/r^2).
\end{aligned}\right\} \tag{25}$$

The numerical coefficients, $\alpha_j \equiv \alpha_{jlsn}, \beta_j \equiv \beta_{jlsn}$, are constants defined by integrals involving triple products of various associated Legendre polynomials and their derivatives. All of these coefficients are defined and evaluated by Munson (1970). The integer N_t determines the order of truncation for the basic flow series used (see part 1).

As shown in part 1 symmetry of the basic flow with respect to the equator gives $f_l(r) = g_j(r) \equiv 0$ for l odd and j even. This, in turn, allows the stability problem to be reduced to two separate eigenvalue problems, (P.1) and (P.2), which can be solved independently; the lowest of the two eigenvalues obtained is the desired solution to the original problem. This reduction to two separate problems provides eigensolutions (disturbance flows) which are symmetric or not symmetric with respect to the equator. Although the problem could be solved without the breakdown into two separate problems, a considerable saving in computer time is obtained via the two problems. Note that for $m \neq 0$ (non-axisymmetric disturbances) the equations are complex (from the $e^{im\phi}$ factor) and must be written in terms of the real and imaginary parts.

As an indication of the structure of these governing equations we consider the following simple example. Assume the basic flow is truncated at the lowest order, $\psi = \sin^2\theta\cos\theta\, g_1(r)$, $\Omega = \sin^2\theta f_0(r)$ (see part 1), that the disturbance flow is truncated with $L_t = 2$,

$$\mathbf{u} = \mathbf{T}_1 + \mathbf{S}_1 + \mathbf{T}_2 + \mathbf{S}_2,$$

and that $m = 0$. Although neither this basic flow truncation nor this disturbance flow truncation are sufficient to obtain meaningful results, the character of the governing equations can be seen. The problem for the eigenfunction components $S_1(r)$, $T_2(r)$ becomes

$$\hat{L}_2^2 T_2 + \rho[(-0{\cdot}333 f_0'/r^2 + 0{\cdot}667 f_0/r^3)\, S_1 + (0{\cdot}428 g_1'/r^2 - 0{\cdot}286 g_1/r^3)\, T_2] = 0,$$

$$\hat{L}_2^4 S_1 + \rho[(-0{\cdot}6 g_1'/r^2 + 0{\cdot}4 g_1/r^3)\, S_1'' + (-0{\cdot}6 g_1''/r^2 + 1{\cdot}6 g_1'/r^3 - 1{\cdot}2 g_1/r^4)\, S_1'$$
$$+ (-0{\cdot}2 g_1'''/r^2 + 0{\cdot}8 g_1''/r^3 - 0{\cdot}8 g_1'/r^4 + 1{\cdot}6 g_1/r^5)\, S_1 + (0{\cdot}6 f_0'/r^2 - 1{\cdot}2 f_0/r^3)\, T_2] = 0,$$

with $S_1 = S_1' = T_2 = 0$ at $r = \eta, r = 1$. A similar problem results for eigenfunction components $T_1(r)$, $S_2(r)$. For higher-order truncations of the basic flow and the disturbance flow, the character of the equations remains the same, but the complexity increases rapidly. Recall that for $m \neq 0$ separate equations for the real and imaginary parts are used.

Thus the task is to solve the system (19), (21), (22) as described above for the minimum eigenvalue, ρ_E.

4. Discussion of stability results

4.1. *Critical Reynolds number*

For a given basic flow situation (that is given radius ratio, η, and angular velocity ratio, $\tilde{\mu}$) we must determine as the solution of the system (19), (21), (22) the curve $\rho = \rho(Re)$ and then the intersection of this curve with $\rho = Re$. This must be done for various wave-numbers, m, and the minimum determined.

With the toroidal–poloidal representation of the basic flow truncated at an appropriate value (see below), the system of equations being considered is a finite set of linear ordinary differential equations with homogeneous two-point boundary conditions and eigenvalue ρ. This problem is solved numerically as an initial-value problem by a forward integration using the Runge–Kutta–Gill method (Harris & Reid 1964, Sparrow 1964). As with the truncation of the basic flow and its numerical solution (see part 1), the numerical solution of the disturbance flow equations becomes 'longer' (in terms of computer storage necessary and computation time required) as the order of the truncation is increased.

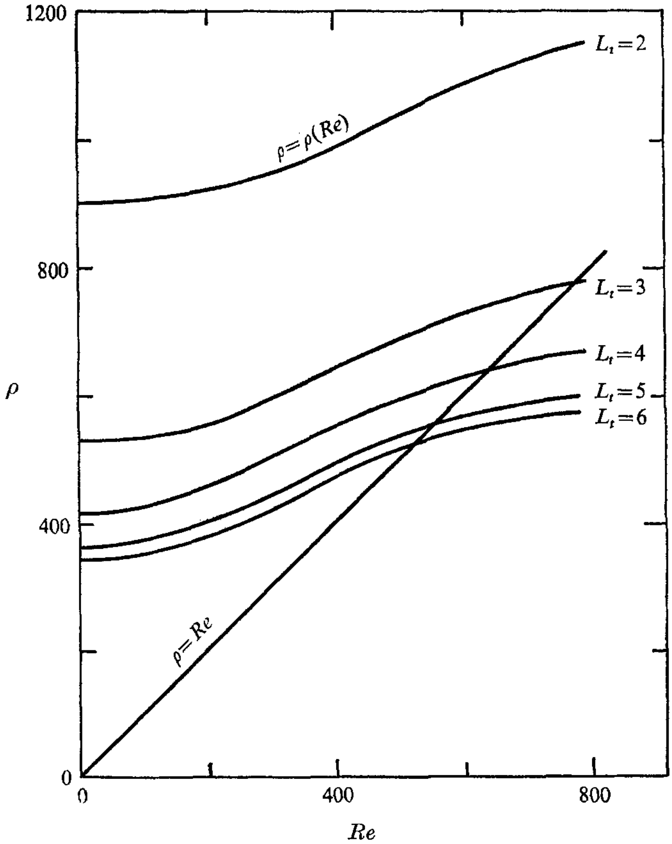

FIGURE 3. Energy eigenvalue, ρ, as a function of basic flow Reynolds number, Re, for various orders of toroidal–poloidal disturbance flow truncation, L_t. $\eta = 0{\cdot}5$, $\tilde{\mu} = 0$, $m = 0$.

Figure 3 shows typical results with $\eta = 0{\cdot}5$, $\tilde{\mu} = 0$, and $m = 0$ for various truncations, $L_t = 2, 3, 4, 5, 6$. The fact that the basic flow character is a function of the Reynolds number shows up in the fact that the $\rho(Re)$ curves are not straight horizontal lines. (For plane Couette flow, for example, the $\rho(Re)$ curve would be a straight line so that a figure such as this would not be necessary.) The convergence to the desired solution as the order of the truncation is increased can be seen in figure 4 where we have plotted the intersection of the $\rho(Re)$ curves with $\rho = Re$ as a function of the order of truncation for various values of the axial wave-number, m. It can be seen that $m = 1$ gives the minimum value of ρ and that the toroidal–poloidal series representation has converged quite well for

$L_t = 7$. These two characteristics were obtained for all six of the basic flow geometries considered.

The critical Reynolds numbers ρ_E (that is, that value below which it is certain that the flow is stable with respect to any type disturbance, large or small) are listed in table 1 for the six cases of the basic flow considered. In spite of the various truncations involved and various inherent numerical errors, etc., it is felt that these results are accurate to at least 5 %. While it is difficult to obtain a 'precise' estimation of this final accuracy, various error indications such as differences with respect to truncation in both the basic flow and the disturbance flow, difference with respect to interval size used in the Runge–Kutta–Gill integration indicate that this estimate of the accuracy is, if anything, conservative.

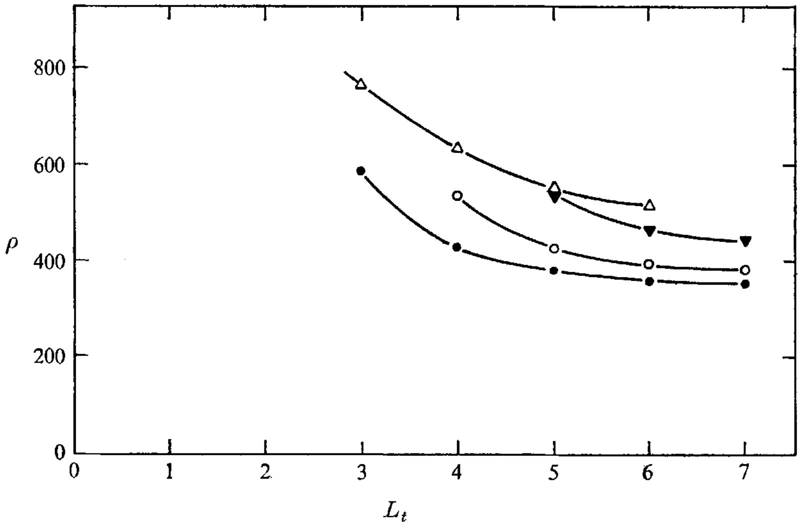

FIGURE 4. Critical Reynolds number, ρ, as a function of toroidal–poloidal disturbance flow truncation, L_t, for various circumferential wave-numbers, m. $\eta = 0{\cdot}5$, $\tilde{\mu} = 0$. Values of m: △, 0; ●, 1; ○, 2; ▼, 3.

Radius ratio $\eta = R_1/R_2$	Angular velocity ratio, $\tilde{\mu} = \Omega_2/\Omega_1$	Critical Reynolds number, ρ_E	Wave-number, m
0·5	∞	$\Omega_2 R_2^2/\nu = 440$ or $\Omega_2 R_2(R_2-R_1)/\nu = 220$	1
0·5	−1	$\Omega_2 R_2^2/\nu = 190$ or $\Omega_2 R_2(R_2-R_1)/\nu = 95$	1
0·5	−0·5	$\Omega_1 R_1^2/\nu = 220$ or $\Omega_1 R_1(R_2-R_1)/\nu = 55$	1
0·5	0	$\Omega_2 R_2^2/\nu = 360$ or $\Omega_2 R_2(R_2-R_1)/\nu = 90$	1
0·75	∞	$\Omega_2 R_2^2/\nu = 760$ or $\Omega_2 R_2(R_2-R_1)/\nu = 190$	1
0·75	0	$\Omega_1 R_1^2/\nu = 690$ or $\Omega_1 R_1(R_2-R_1)/\nu = 130$	1

TABLE 1. Critical Reynolds number for flow between rotating spheres

In all of the previous discussion (both basic flow and stability consideration), the Reynolds number has been defined with R_2 as the characteristic length and $\Omega_0 R_2$ (where Ω_0 is either Ω_1 or Ω_2) as the characteristic speed. This was done to avoid confusion and to make the discussion less cumbersome. However, it is of interest now to use a more appropriate definition of Re. That is to let $(R_2 - R_1)$, the gap size, be the characteristic length and $\Omega_j R_j$, $j = 1$ or 2, be the characteristic speed, depending on which sphere is the 'dominant' one. The critical Reynolds numbers with these new definitions are shown in the above table also.

Not unexpectedly, the case where the inner sphere is stationary and the outer sphere rotates ($\tilde{\mu} = \infty$) is the most stable, and rotation of the inner sphere causes a reduction in the stability. Although there is a difference between the $\eta = 0{\cdot}5$ and $\eta = 0{\cdot}75$ cases, it is not great. Consideration of figures 4 to 7 in part 1 will provide an idea of the character of the basic flows whose critical Reynolds numbers are given in table 1.

It is noted that Payne (1964) has given a rigorous estimate of the Reynolds number below which the motion is surely stable. This estimate, based on boundary data and not the entire basic flow solution, is necessarily quite conservative and gives $\Omega_2 R_2^2/\nu = 0{\cdot}26$ rather than the value of 440 shown in table 1 for the case with $\eta = 0{\cdot}5$, $\tilde{\mu} = \infty$.

4.2. *Disturbance flow characteristics*

Along with the critical Reynolds number, it is of interest to determine the disturbance flow field that is obtained for the lowest eigenvalue. We consider the disturbance flow for the case where $\eta = 0{\cdot}5$, $\tilde{\mu} = 0$; the other cases have a similar character. With $m = 1$ (the axial wave-number that gives the lowest eigenvalue) and the toroidal–poloidal representation truncated at $L_t = 7$, we can write the disturbance velocity components, (u, v, w), as follows from (14):

$$\left.\begin{aligned} u &= \sum_{\substack{l=2\\ \text{even}}}^{6} \frac{l(l+1)}{r^2} P_l^1 [S_l^r \cos\phi - S_l^i \sin\phi], \\ v &= \sum_{\substack{l=2\\ \text{even}}}^{6} \frac{1}{r} \dot{P}_l^1 [S_l^{r\prime} \cos\phi - S_l^{i\prime} \sin\phi] - \sum_{\substack{l=1\\ \text{odd}}}^{7} \frac{1}{r\sin\theta} P_l^1 [T_l^r \sin\phi + T_l^i \cos\phi], \\ w &= \sum_{\substack{l=2\\ \text{even}}}^{6} -\frac{1}{r\sin\theta} P_l^1 [S_l^{r\prime} \sin\phi + S_l^{i\prime} \cos\phi] - \sum_{\substack{l=1\\ \text{odd}}}^{7} \frac{1}{r} \dot{P}_l^1 [T_l^r \cos\phi - T_l^i \sin\phi], \end{aligned}\right\} \tag{26}$$

where $S_l \equiv S_l^r + iS_l^i$ and $T_l \equiv T_l^r + iT_l^i$. Only S_j, T_l with l odd, j even are considered (the others being identically zero) because, it turns out, the problem (P. 2) gives the minimum eigenvalue. This implies that the disturbance field is not symmetric with respect to the equator, as is the basic flow, but from the fact that

$$\left.\begin{aligned} u(r,\theta,\phi) &= -u(r,\pi-\theta,\phi), \\ v(r,\theta,\phi) &= v(r,\pi-\theta,\phi) \\ w(r,\theta,\phi) &= -w(r,\pi-\theta,\phi) \end{aligned}\right\} \tag{27}$$

we need only consider the northern hemisphere, for example.

It is very difficult to picture the three-dimensional disturbance flow field because it is a function of all three spatial co-ordinates, unlike the basic flow which is independent of ϕ and allows the introduction of the stream function in the meridian plane. Also the disturbance flow does not take on what may be called a 'completely spherical character', so that precise picturing of the flow is difficult. It is possible, of course, to calculate particle paths as given by the disturbance velocity, but picturing these paths is difficult also. Also the fact that the actual particle path results from a superposition of the disturbance velocity (of unknown magnitude) on to the already complicated basic flow velocity makes such a procedure meaningless.

However, there are two distinct characteristics of the disturbance flow which, if this energy disturbance were the actual physical disturbance, would be clearly distinguishable from the basic flow. These are that the disturbances consist of flows across both the equatorial plane and the pole. The fluid flows horizontally across the polar region in a specific ϕ direction and vertically across the equator in an exchange of fluid between the northern and southern hemispheres. It should be noted that this flow across the poles is obtained only because $m = 1$ gives the critical wave-number. For any other value of m it can be shown that $u, v \rightarrow 0$ as $\theta \rightarrow 0$ (since, for $m \neq 1$, $P_l^m(\cos\theta) \rightarrow 0$ as $\theta \rightarrow 0$) and there can be no flow across the poles.

Although the disturbance flow field is very complex the basic character can be described as: (*a*) a horizontal flow across the poles, (*b*) two complex swirl type patterns near the equator and opposite each other longitudinally, and (*c*) a vertical flow across the equatorial plane. Further details are provided by Munson (1970).

As mentioned previously this complex disturbance flow is superimposed on to the basic flow, but the distinctive character of the disturbance flow near the equator and the poles should be observable in an experiment, provided the energy disturbances are obtained physically.

5. Linear theory

The above discussion has dealt with the energy (stability) theory for the flow between concentric spheres. It is of considerable interest to likewise consider the linear (instability) theory for the same problem. In general, the two limits (energy and linear) do not coincide – the resulting gap between the two being a region where possible sublinear instabilities may be present. As seen in Joseph & Munson (1970), where the stability of a viscous fluid between rotating-sliding cylinders is considered, the effect of rotation can, in many instances, drive the energy and linear limits into coincidence or near coincidence. Although the linear theory has not been calculated, it would not be unexpected for the linear and energy results to be close, at least for certain cases.

The only previous consideration of the linear theory for flow between rotating spheres is that of Bratukhin (1961). He has considered an approximate solution with axially symmetric disturbances ($m = 0$) and with the basic flow given by the Stokes flow, or first-order (low Reynolds number) perturbation solution. He considered the case with $\eta = 0{\cdot}5$ and $\tilde{\mu} = 0$ and obtained a critical Reynolds number of approximately 100; $\Omega_1 R_2^2/\nu \approx 400$ or $\Omega_1 R_1(R_2 - R_1)/\nu \approx 100$. His disturbance flow is axially symmetric (since he only considered $m = 0$) and symmetric with respect to the equator – thus very similar to the basic flow pattern and unlike our energy disturbances. Although as Bratukhin says this value of 100 may not be very accurate (only axially symmetric disturbances are considered, the basic flow is not accurately given by the first-order perturbation solution, and his solution of the stability problem is a series truncated at second order), it, nevertheless, indicates that the energy and linear limits for the flow between rotating spheres may be very nearly the same, perhaps identical under some conditions. For example, Bratukhin's approximate linear result of $\Omega_1 R_1(R_2 - R_1)/\nu \approx 100$ for $\eta = 0{\cdot}5$, $\tilde{\mu} = 0$ is quite near the energy result of 90 for this case (see table 1). A thorough investigation of the linear stability problem would be of considerable interest.

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